Causal boundaries and holography on wave type spacetimes

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Abstract. The notion of *causal boundary* for a spacetime has been a controversial topic during the last three decades. Moreover, recently the role of the boundary in the AdS/CFT correspondence for plane waves, have stimulated its redefinition with some possible alternatives.

Our aim is threefold. First, to review the different classical approaches to boundaries of spacetimes, emphasizing their drawbacks and the progressive redefinitions of the c-boundary. Second, to explain how plane waves and AdS/CFT correspondence come into play, stressing the role of the c-boundary as the holographic boundary in this correspondence. And, third, to discuss the present-day status of the c-boundary, making clear the arguments of a definitive proposal.

Keywords: causal boundary, Geroch, Kronheimer and Penrose (GKP) precompletion; conformal boundary; Budic and Sachs boundary; Szabados relation; future chronological boundary; Harris universal properties; Marolf and Ross pairs; Flores chronological completion; causal structure; pp-waves; Mp-waves, plane wave string backgrounds; Penrose limit; AdS/CFT; Berenstein, Maldacena and Nastase correspondence.

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Contents

1 Introduction

In the framework of Mathematical Relativity, the causal boundary, or c-boundary, is an appealing construction proposed initially by Geroch, Kronheimer and Penrose [24], in order to attach a conformally invariant boundary ∂M to any reasonable spacetime $M(\equiv (M,g))$. Roughly, its purpose is to attach a boundary endpoint P to any inextensible future or past directed timelike curve γ, and the original basic idea is simple: the boundary point would be represented by the past or the future of the curve, $P = I^{\pm}(\gamma)$. By construction, the c-boundary ∂M is conformally invariant, and does not contain any direct information on singularities, which may depend strongly on the conformal factor –∂M aims to model "points at infinity" or even "conformally invariant singularities". Nevertheless, the purely conformal information contained in ∂M may yield a pleasing picture of the spacetime. Say, in principle, ∂M would be the union of three disjoint subsets: the *future infinity* $\partial_+ M$, reached by future-directed timelike curves but no past-directed ones, the *past infinity* $\partial_- M$, dual to the former, and the *timelike boundary* $\partial_0 M$, whose points are reached by both, future and past directed curves. The latter may represent naked singularities, the boundary of a removed region in a bigger spacetime or, in general, losses of global hyperbolicity, i.e., points where $J(p,q) := J^+(p) \cap J^-(q)$ is not compact[1]. Eventually, asymptotically conformally flat regions would be represented by "lightlike parts" in $\partial_\pm M$. Further properties may yield information on black holes or other conformally invariant properties. In an ideal scenario, initial conditions for evolution equations would be a sort of limit in $\partial_- M$, while boundary conditions would be posed on $\partial_0 M$.

However, a satisfactory notion of c-boundary must comprise not only the definition of ∂M but also the extension of both, the topology and the chronological relation to the completion $\bar{M} = M \cup \partial M$. So, $\bar{M}$ must satisfy strong requirements in order to be regarded as satisfactory, and many specialists have been puzzled along the last three decades with them. Concretely, with the three apparently harmless questions:

Q1 *Point set definition.* The construction of the c-boundary ∂M provides automatically identifications between the boundary points attached to different (say) future-directed curves with the same past. Nevertheless, in order to obtain a satisfactory c-boundary, some inextensible timelike curves with *opposed time orientations* must also determine the same boundary points (Fig. 1). And a suitable prescription to solve such *identification problem* is not by any means trivial.

Q2 *Chronology.* Being the role of Causality obvious for ∂M, causal relations $\ll, <$ must be extended to the boundary. Such extended relations $\overline{\ll}, \overline{<}$ would also fulfill natural requirements, as being transitive and equal to the original ones $\ll, <$ on M. Nevertheless, these simple requirements are not so easy to fulfill, specially when combined with others (notice that, if $Q \in \partial M$ satisfies $x \overline{<} Q \overline{\ll} y$ for some $x, y \in M$, this may yield a new –spurious– relation $x \overline{<} y$ or even $x \overline{\ll} y$, which did not exist for $<, \ll$ in M).

Q3 *Topology.* The appropriate topologies for the boundary ∂M and the completed space $\bar{M} = M \cup \partial M$, should fulfill some natural requirements –for example, the inclusion $i : M \hookrightarrow \bar{M}$ must be a topological embedding and $i(M)$ a dense subset in $\bar{M}$. But subtler questions appear, because there are many simple examples (say, open subsets of Lorentz-Minkowski

[1] Global hyperbolicity will be violated because of these points, whenever M is causal, [7].

$\mathbb{L}^n$) where both, the c-boundary as a point set and its topology, seem obvious. It is not clear at what extent the c-boundary (a conformally invariant object) must recover such an "obvious boundary". Moreover, there is no a clear rule about which separation properties must satisfy the boundary points –among them, and with points in M.

Relevant contributions by many authors such as Budic and Sachs [9], Rácz [47, 48], Szabados [54, 55], Kuang and Liang [37, 38] or Harris [26, 27, 28], illuminated both, the possible definitions of c-boundary and its limitations. Nevertheless, the notion of c-boundary was unclear at the beginning of the 21th century, and some authors claimed the impossibility of a natural general definition.

The interest in the c-boundary has been renewed in the last years, because of the advances in the holographic principle and AdS/CFT correspondence. According to this conjectured correspondence, a string theory on a background spacetime becomes equivalent to a field theory (the hologram) on its boundary. The question relevant to us, is which holographic boundary must be chosen. Typically, the Penrose *conformal boundary* is used, as this is the most common boundary in Mathematical Relativity. Nevertheless, the detailed AdS/CFT correspondence for plane wave backgrounds [4, 5] stresses the limitations of this boundary. This led Marolf and Ross [40] to consider the c-boundary instead of the conformal one. In fact, taking into account the progress of previous authors, they redefined completely the boundary [41]. Remarkably, they introduced the idea of boundary points as pairs (P, F) of certain past and future sets. Under this viewpoint, they emphasize that a chronology $\overline{\ll}$ on the completion suggested by Szabados, turns out consistent now, and is the unique natural chronology. They also obtained interesting applications of the c-boundary to the holography of plane waves, which were independent of the concrete details of their approximation [40]. Moreover, such results were reobtained and widely extended for the general framework of wave-type spacetimes in [17]. But, in spite of this very big progress, Marolf and Ross breakthrough could not be regarded as definitive yet. The question was obvious: taking into account the failure of all previous approaches, to what extent their choices were unique or undisputable? For example, in the definition of the pairs $(P, F) \in \partial M$, Marolf and Ross used Szabados relation $\sim_S$, which modified a previous one by Budic and Sachs, is this choice unavoidable? Notice that they introduced two alternative topologies, and suggested the possible existence of an intermediate topology with better properties.

Recently, Flores revisited systematically the definition of the c-boundary [13]. Admitting the Marolf-Ross viewpoint that boundary points must be regarded as pairs (P, F) of some past and future sets, he introduced a very general notion of *completion* for a spacetime, as well as a natural topology, inspired in previous work by Harris. Among all the possible completions, the *minimal –chronological–* completions become a privileged (non-empty) subfamily. Then, a detailed study of the properties satisfied by any completion, is carried out. In particular, minimal completions satisfy a very reasonable set of satisfactory properties, required in principle for any c-boundary (both, as a point set and topologically). The problem of minimal completions is that, in general, they are not unique. Even though this problem is not as bad as it sounds –Flores emphasized the role of chronological completions, and the properties satisfied by them might be enough for many purposes–, his study goes further. In general, Marolf-Ross completion is not minimal, but that completion: (i) shares all the other good properties of the minimal ones, and (ii) satisfies that any minimal completion can be naturally included in it. That is, Marolf-Ross completion is now univocally singled out as a point-set. Moreover, the properties of Flores topology proved in [13],

plus others to be discussed here (announcing also results in [16]), single out this topology too. So, Marolf-Ross point-set completion, endowed with Flores topology, and the natural chronology $\overline{\ll}$, are selected. This choice will be called the *Flores boundary* here, in order to distinguish it from other possible alternatives. However, the main aim of the present paper is to emphasize that this must be regarded as the genuine c-boundary, constructed after the works of all previous authors.

The present article is organized as follows. Some brief comments on other boundaries, different to the causal one, are given in Section 2, and their own drawbacks are also pointed out. The old different approaches for the notion of c-boundary are reviewed in Section 3, especially, questions Q1, Q2, Q3, are stressed. The roles of holography and plane waves are emphasized in Section 4. In Section 5 both, Marolf and Ross [40, 41] and Flores [13] approaches are explained, and reasons for our definitive proposal are discussed. Some of the arguments will be developed further in [15, 16]. In Section 6 we sketch briefly the results and involved techniques for the boundary of the wave-type spacetimes, following [17]. This leads to a highly non-trivial problem because, in general, the explicit computation of causal boundaries (TIPs, TIFs) requires new techniques: Busemann functions, variational interpretations or specific tools for some type of concrete spacetimes. We point out that the c-boundary of wave-type spacetimes requires a combination of Busemann-type functions, variational methods and Sturm-Liouville theory. Finally, in Section 7, we include a brief summary of our proposal of c-boundary, for the convenience of the reader.

2 Preliminaries: some type of boundaries

In the following, we will consider standard notation and conventions as in the classical references [3, 31, 44, 56] or in the recent reviews [22, 43]. In particular, any Lorentzian manifold (M, g) will have signature $(-, +, \ldots, +)$, any spacetime M is a (connected) time-oriented Lorentzian m-manifold, where the time orientation is assumed implicitly, and causal vectors are distributed in two cones, each one containing future or past-directed timelike ($g(v, v) < 0$, $v \in TM$) and lightlike ($g(v, v) = 0$, $v \neq 0$) tangent vectors.

First of all we review some types of boundary used in General Relativity, different to the causal one. All of them appear under a natural viewpoint. Nevertheless, we stress that none of them is fully satisfactory. This will stimulate the efforts to overcome the difficulties for the c-boundary.

2.0.1 Penrose conformal boundary.

In Riemannian Geometry, the stereographic projection yields a natural conformal open embedding $i : \mathbb{R}^n \hookrightarrow \mathbb{S}^n$ of Euclidean space $\mathbb{R}^n$ into the sphere $\mathbb{S}^n$, being the boundary of the image $\partial i(\mathbb{R}^n) \subset \mathbb{S}^n$ a point which may be interpreted as the conformal infinity of $\mathbb{R}^n$. In Lorentzian Geometry there exists also a natural conformal open embedding $i_P : \mathbb{L}^n \hookrightarrow \mathbb{L}^1 \times \mathbb{S}^{n-1}$ of Lorentz-Minkowski $\mathbb{L}^n$ in the Einstein static Universe $\mathbb{L}^1 \times \mathbb{S}^{n-1}$. As pointed out by Penrose, here the boundary $\partial i_P(\mathbb{L}^n) \subset \mathbb{L}^1 \times \mathbb{S}^{n-1}$ is compact and contains certain elements with natural interpretations: the point i^0 or spacelike infinity, the points $i^\pm$ or timelike infinities and the null hypersurfaces $\mathcal{J}^\pm$ or null infinity (see for example, [31, 56]). This embedding suggests the definition of *asymptotically flat spacetime (at null or spacelike infinity)*, as a spacetime which admits an open conformal embedding in a bigger "aphysical" spacetime $\tilde{M}$ (necessarily of the same dimension), which qualitatively behaves as i_P close to the corresponding elements at infinity $\mathcal{J}^\pm, i^0$.

This notion has been widely used in General Relativity: isolated bodies, mass, or black holes are naturally modelled in asymptotically flat spacetimes. Nevertheless, the technical conditions which define an asymptotically flat embedding are very specific (see [56, Sect. 11.1] for a discussion). These conditions imply the essential uniqueness of the asymptotic part of the boundary (so, this part of the boundary can be regarded as intrinsic, [2]). But the limitations of the approach become obvious. For a general spacetime M (under, say, some global reasonable assumption, as being strongly causal or even stably causal), one can try to find a conformal embedding $i : M \hookrightarrow \tilde{M}$ such that (i) $i(M)$ is an open subset of the "aphysical" spacetime $\tilde{M}$ (i.e. dim M =dim $\tilde{M}$) and (ii) the closure of $i(M)$ in $\tilde{M}$ is compact. Then, the boundary $\partial i(M) \subset \tilde{M}$ can be regarded as a sort of conformal boundary. But, in general, this is neither intrinsic (depends on the embedding) nor systematic (there is no a way to determine if such conformal embedding exists and, in this case, how to construct one). In fact, one can check that a part of $\partial i(M)$ can be regarded as the causal boundary (even though perhaps with some points artificially identified). Because of this reason only this part becomes truly intrinsic and systematic, but even in this case the topology is not canonical –this will be developed in forthcoming [16].

Further developments in related directions have been carried out. García-Parrado and Senovilla introduced *isocausal extensions*, which also yield a conformally invariant boundary [21] –see also [20]. Such extensions are less rigid (and easier to find) than conformal extensions, even though by this reason their lack of uniqueness is also bigger. Scott and Szekeres introduced the *abstract boundary* [46]. Even though their main aim was to study singularities, this boundary is general, and is defined by using open embeddings (*envelopments*) of the underlying *differentiable manifold*. The so obtained a-boundary is unique for the manifold –as all the possible envelopments are considered. When one focuses in concrete classes of curves (for example, affinely parametrized geodesics for a semi-Riemannian metric), some other boundaries are redefined under this different framework. However, neither the topology of the a-boundary nor the framework of the causal boundary were studied in the original article. Even though some development on the topology was obtained in [12], a further study would be interesting.

2.0.2 Geodesic and bundle boundaries

Geroch's g-boundary [23] and Schmidt's b-boundary [51] are constructed in order to deal with singularities, so, they may look very different to conformal or c-boundaries. The g-boundary is defined in terms of classes of incomplete geodesics, and Geroch explored several possibilities for these classes —the weakest identifications yield a T_0 topology for the quotient. The b-boundary is a mathematically elegant construction, obtained by defining a certain positive definite metric on the bundle of linear frames LM of any semi-Riemannian manifold (M, g); then, the Cauchy completion of LM induces the b-boundary for (M, g).

Both constructions are systematic, non-conformally invariant, and satisfy the following *a priori desirable* properties :

(i) every incomplete geodesic in the original spacetime terminates at a point[2],

(ii) they are geodesically continuous, in a well defined sense.

[2]According to Scott and Szekeres [46], a compact manifold should not have neither singularities nor a boundary. So, as compact Lorentzian manifolds may be incomplete, this criterium would not be fulfilled by any boundary satisfying this a priori desirable condition.

The b-boundary was known to give unphysical results for common solutions to Einstein's equations. For example, the extended Schwarzschild spacetime M contains a b-boundary point i such that every neighborhood of i contains all M –and no clear alternative to Schmidt's construction seemed to exist [52].

Moreover, a remarkable drawback was found for any construction satisfying the desirable conditions above: a simple spacetime (M, g) constructed by Geroch, Liang and Wald [25] shows that minimal conditions (i), (ii) yield a rather undesirable topology. The example is relatively easy to construct (the spacetime is just $M = (\mathbb{R}^2 \backslash \{s\}) \times \mathbb{R}^2$, $s \in \mathbb{R}^2$, and the metric $g = \Omega\langle\cdot,\cdot\rangle_{\mathbb{L}^2} + \langle\cdot,\cdot\rangle_{\mathbb{R}^2}$, for some appropriate function $\Omega > 0$; a variant of the example is even flat), and the a priori undesirable property of the boundary is the existence of a boundary point s not T_1 related to some point in the spacetime M.

Of course, such an example does not affect the mathematical validity of the boundaries. So, its true importance, as well as related constructions of the boundaries, may deserve a further study (recall, for example, the space of lightlike geodesics [39] and its relation to linking problem [11]). But, at any case, such a problem will not affect the causal boundary.

3 Causal boundaries prior to the holography of waves

As commented in the Introduction, the purpose of c-boundaries is to attach a point to any inextensible timelike curve of some spacetime M. As both, the topology and chronological relation will be extended to the completed spacetime $\bar{M}$, some mild causality conditions on M will be used as: *chronology* (inexistence of closed timelike curves), *causality* (inexistence of closed causal curves), to be *(past or future) distinguishing* (future distinguishing: different $p, q \in M$ have different $I^+(p), I^+(q)$; past distinguishing: analogous with $I^-(p), I^-(q)$; distinguishing: future and past distinguishing) or *strong causality* (inexistence of "almost closed" causal curves). Recall that strong causality is also equivalent to the equality between the natural topology of M and its Alexandrov topology, i.e., the one generated by $I^{\pm}(p)$ for all $p \in M$.

3.1 Starting c-boundaries: GKP construction

The Geroch, Kronheimer and Penrose [24] boundary is a general construction, in principle applicable to any strongly causal spacetime M, and it is explicitly intrinsic, systematic and unique.

The construction has a very appealing first part: the definition of the *precompletion* $M^\sharp$, which, essentially, is retained in all the subsequent developments of the c-boundary. Briefly (see for example [30] in this proceedings for more details), one starts by declaring that $P \subset M$ is a past set if $P = I^-(P)$ and, then, P is an IP (indecomposable past set) if it cannot be written as the union of another two past subsets. Such an IP is necessarily either *proper* (PIP) or *terminal* (TIP). In the former case, $P = I^-(p)$ for some $p \in M$ (so, M itself is identifiable to the set of all PIP's, as the spacetime is past distinguishing), in the latter, $P = I^-(\gamma)$ for some inextendible future-directed timelike curve

The set of all TIP's is the *future preboundary* $\hat{\partial} M$ of M, and the set of all the IP's is the future precompletion $\hat{M}$. Analogously, indecomposable future sets, (IF's), which may be either proper (PIF's) or terminal (TIF's) are considered, and one defines the past preboundary $\check{\partial} M$ and past

precompletion $\check{M}$. The precompletion $M^\sharp$ of M is essentially M plus the preboundary points or, more precisely: $M^\sharp = (\hat{M} \cup \check{M})/\sim$ where $\sim$ is the relation of equivalence $I^+(p) \sim I^-(p), \forall p \in M$.

However, the next steps in the GKP construction concerns the questions Q1, Q3 in the Introduction (identifications and topology), and they became widely controversial. The GKP construction tries to solve both questions at the same time. First $M^\sharp$ is topologized by taking as a subbase the sets F^{int}, F^{ext}, P^{int}, P^{ext}:

$$F^{int} = \{P \in \hat{M} : P \cap F \neq \emptyset\}, \quad , \quad F^{ext} = \{P \in \hat{M} : P = I^-[\omega] \Rightarrow I^+[\omega] \not\subset F\}, \qquad \forall F \in \check{M},$$

and analogously for $P \in \hat{M}$. Notice that, when F is a PIF, $F = I^+(p)$, then the set of PIP's corresponding to F^{int} is equal to F itself, and the set corresponding to F^{ext} is $M \backslash \bar{J}^+(p)$. In this sense, this topology is Alexandrov-type. As pointed out in [24, Figure 6], the sets type P^{ext}, F^{ext} are required to give a reasonable basis for the topology in the possible "lightlike part" of the boundary. Finally, the causal completion would be defined as the quotient $\bar{M} = M^\sharp/R_H$, where R_H is the minimum identification of *preboundary points* to obtain a Hausdorff space.

This *a priori* imposition of Hausdorffness was found unsatisfactory by several authors. Perhaps, the most surprising drawback was pointed out by Kiang and Lian [37]: the GKP boundary recovers well the natural boundary of a timelike half plane of $\mathbb{L}^2$, but this is not the case for a half space of $\mathbb{L}^3$ (Figs. 1 and 2). However, this was not the unique problem:

(1) One such identification only between preboundary points yielding a Hausdorff quotient (and, then, the intersection of all them, R_H) may not exist in general, see [54]. Intuitively, the reason is that strong causality might be lost "at a boundary point" and, so, one such point might be non-Hausdorff related to points of the spacetime. Nevertheless, such an identification does exist if the spacetime is stably causal; so, the approach must be restricted to this class of spacetimes.

(2) Taub spacetime is static, but its GKP "singularity" is only a point, not a line [36], as one would expect –unsatisfactory properties for plane waves under GKP and other approaches will be also found [41, Section 5].

(3) For $\mathbb{L}^n$, the topology of the (causal part of the) Penrose conformal boundary does not agree with the GKP topology [27][3].

3.2 First modifications

Budic and Sachs [9], Rácz [47] and Szabados [54, 55], proposed some modifications of the GKP construction, in order to overcome previous problems. Let us start with the second one.

3.2.1 Rácz's topology

The simplest modification is to change the "Alexandrov type" topology. Rácz proposed the modifications of the subsets F^{int}, F^{ext} (and analogously for P). Essentially: (i) these open sets are defined only when F is a PIF (no a TIF), and (ii) they generate not only IPs but also IFs . More precisely:

$$F^{int} = \{A \in \hat{M} \cup \check{M} : A \in \hat{M} \text{ and } A \cap F \neq \emptyset \text{ or } A \in \check{M} \text{ and } A = I^+[S] \Rightarrow I^-[S] \cap F \neq \emptyset\},$$

[3] The boundary $\hat{\partial}\mathbb{L}^n$ is a cone, as in Penrose conformal embedding. But each one of its null generators is a GKP-open subset, by arguments similar to those in Fig. 2. Such a problem will appear also in other approaches which essentially maintain the GKP topology (see also [14]).

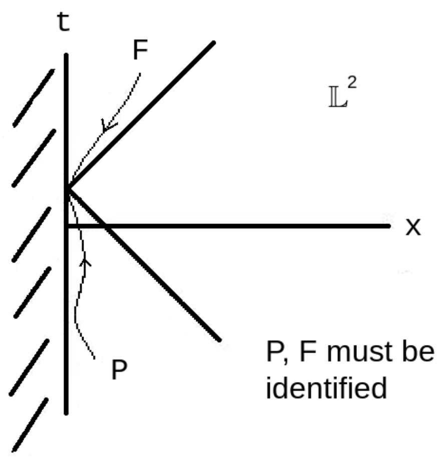

Figure 1: Spacetime $\mathbb{L}^1 \times \mathbb{R}^+ \subset \mathbb{L}^2$ $(x > 0)$. The relation of equivalence R_H must be introduced in order to recover the natural boundary: as $P = I^-(\rho), F = I^+(\gamma)$ "finish" at the same point with $x = 0$, they have to be identified to a single point Q. This happens in the GKP topology as P, F are T_1-related ($P \in F^{ext}$, $F \in P^{ext}$) but not T_2-related in $M^\sharp$ (say, $P^{ext} \cap F^{ext} \neq \emptyset$).

$$F^{ext} = \{A \in \hat{M} \cup \check{M} : A \in \check{M} \text{ and } A \not\subset F \text{ or } A \in \hat{M} \text{ and } A = I^-[S] \Rightarrow I^+[S] \not\subset F\},$$

Some problems of the GKP construction do not appear in Rácz one (for example, the boundary of Taub is 1-dimensional –but again the spacetime must be stably causal, in order to ensure the existence of the relation of equivalence R_H). However, Kuang and Liang [38] found an example unfavorable to Rácz topology (Fig. 3).

Notice that Rácz topology on $M^\sharp$ maintains the GKP relation $P \in F^{ext}$ pointed out in Fig. 1. In principle, this behavior seems rather artificial, but becomes essential in order to ensure that P, F are not T_2 related –and, thus must be identified with the same point.

3.2.2 Budic and Sachs' equivalence relation

Budic and Sachs made a full revision of the GKP construction. One of the main ingredients they introduced (which will turn out to be essential for our final proposal) is a direct identification between TIPs and TIFs –recall that both, GKP and Rácz identifications were defined indirectly. Namely, for $P \in \hat{M}, F \in \check{M}$:

$$P \sim_{BS} F \quad \Longleftrightarrow \quad P = \downarrow F \text{ and } F = \uparrow P$$

where, say, the common past $\downarrow F$ is the interior of $\cap_{x \in F} I^-(x)$. Then, they also extended both chronological and causal relations on the quotient space $\overline{M} = M^\sharp / \sim_{BS}$. By using these relations, an Alexandrov-type topology was defined on $\bar{M}$. The construction seemed specially good for *causally continuous* spacetimes, but Kuang and Liang [37] also found a unfavorable example in

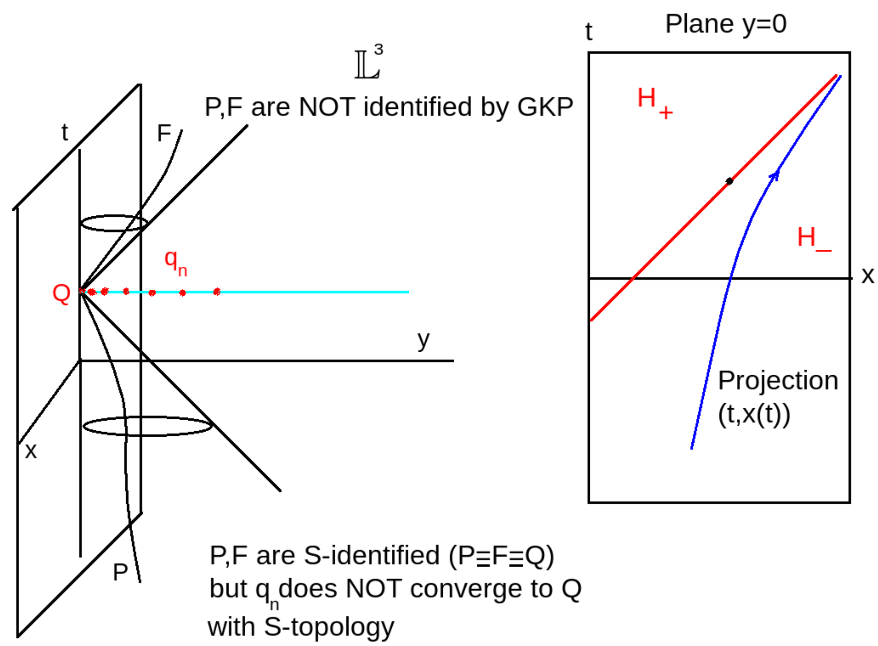

Figure 2: GKP construction does not recover the natural boundary of $\mathbb{L}^2 \times \mathbb{R}^+ \subset \mathbb{L}^3$ ($y > 0$): $P = I^-(\rho), F = I^+(\gamma)$ are not identified as they are T_2-related. In fact, a lightlike plane which intersects orthogonally the (conformal) boundary $y = 0$ at Q yields two hemi-spaces $H_\pm$. Now, $H_- = I^-(\alpha)$ for some timelike $\alpha(t)$ (type $\alpha(t) = (t, x(t), (1+t^2)^{-1})$). So, H_- is a TIP, H_+ a TIF, and $H_\pm^{ext}$ separates P and F ($P \in H_+^{ext}$, $F \in H_-^{ext}$ and $H_+^{ext} \cap H_+^{int} = \emptyset$), [37]. Szabados relation does identify P, F to the single point Q, but $\{q_n\}_n \not\to Q$ with his topology [38].

this case (Fig. 3). Intuitively, the BS approach is conceived for spacetimes M^m with no "$(m-1)$-dimensional" parts removed, or "m-dimensional holes". In fact, if one removes a $m-1$ dimensional part, undesirable properties may appear, see Fig. 4. Nevertheless, causal continuity cannot prevent the existence of holes, as Fig. 3 shows. Moreover, analogous problems for causally simple spacetimes were also pointed out by Rübe [49]. The problems of the identifications become then specially delicate, as causal simplicity is the step in the standard causal hierarchy of spacetimes which follows to causal continuity, and the next step is global hyperbolicity —where the problem becomes trivial, as no identifications can appear.

3.2.3 Szabados reformulation

Szabados made a penetrating study on which points of the GKP-boundary can be separated from points of the spacetime. As stable causality or causal continuity do not prevent the appearance of undesirable situations, he tried to give a solution for any strongly causal spacetime.

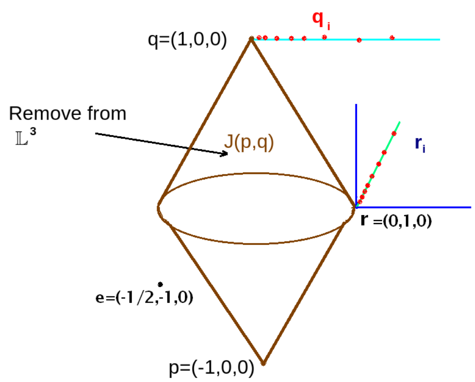

Figure 3: Causally continuous spacetime $M = \mathbb{L}^3 \backslash (J^+(t=-1, x=0, y=0) \cap J^-(1,0,0))$. The removed points $r = (0,1,0), q = (1,0,0)$ are naturally identified with boundary points. Nevertheless: (a) for Rácz topology, $\{r_n = (0,1,1/n)\} \not\to r$, as $I^+(-1,2,0)^{int} \cup I^-(1,2,0)^{int}$ projects onto an open subset of the completed space which contains r but no r_n [38], (b) for BS completion, $\{q_n = (1,1/n,0)\} \not\to q$, as $e = (-1/2,-1,0)$ is not chronologically related to any q_n, but $e \ll q$ in the completed spacetime $\bar{M}$ –that is, $I^+(e)$ yields an open subset of $\bar{M}$ containing q but no q_n.

Szabados reformulated the BS identification in $M^\sharp$, in order to deal also with examples as in Fig. 4. In the set of all the IPs and IFs $\hat{M} \cup \check{M}$, he introduces the relations:

$$P \sim_S F \Longleftrightarrow \left\{ \begin{array}{lll} F & \text{is included and is maximal in} & \uparrow P \\ P & \text{is included and is maximal in} & \downarrow F \end{array} \right. \tag{3.1}$$

and extend this by transitivity to a relation of equivalence (here "maximal" means with respect to the partial ordering $\subset$). Moreover, he also introduced a second technical identification. This involves terminal indecomposable sets of the same type (TIP-TIP or TIF-TIF), as sometimes this becomes natural (see Fig. 5).

This defines the boundary as a point set. A chronological relation and a topology were also introduced. Even though the chronological relation had some problems inherent to the identification approach pointed out by Marolf and Ross (see Example 5.7(3) below), it will suggest the final choice of the chronological relation. About the topology, some previous problems of the GKP boundary do not appear, but Kuang and Liang [38] found a new very unfavorable example. In fact, the Szabados construction does not recover well the topology for the completion of $\mathbb{L}^2 \times \mathbb{R}^+$, Fig. 2.

What is worse, the Kuang and Liang example will hold whenever: (i) the GKP topology is used and (ii) the identifications rule is such that a TIF $F = I^+(\lambda)$ and a TIP $P = I^-(\gamma)$ corresponding to the boundary of the removed region $y \leq 0$ are identified iff λ, γ have the same point $(t_0, x_0, 0)$ as their endpoint. Therefore, these authors claimed [38]:

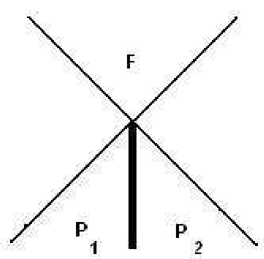

Figure 4: Because of the removed segment of $\mathbb{L}^2$ $\downarrow F = P_1 \cup P_2$. Thus $P_i \not\sim_{BS} F$ for $i = 1, 2$. As $P_1 \sim_S F \sim_S P_2$, Szabados imposes that the three sets represent a single boundary point. Marolf-Ross will take both pairs (P_i, F) as distinct boundary points.

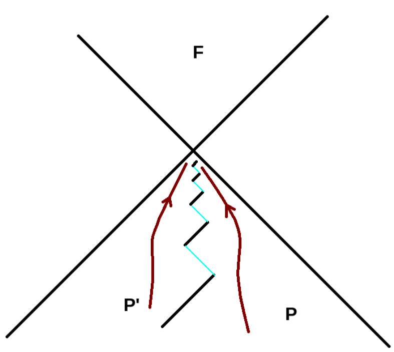

Figure 5: In $\mathbb{L}^2$ minus a removed sequence of thick segments (at 45 degrees) one has: $P' \subsetneq P$, $F = \uparrow P = \uparrow P'$; thus $P' \not\sim_S F$. A second Szabados relation identifies P and P'. Marolf-Ross take both pairs $(P, F), (P', \emptyset)$ and their topology is not T_1 here.

We are inclined to believe that the whole project of constructing a singular boundary has to be given up.

Even though there were reasons for this pessimistic viewpoint[4], the analysis of previous approaches makes reasonably clear: (a) GKP identification rule must be abandoned, (b) GKP topology (and its naive variations) have to be changed, and (c) one cannot forget to find a good extension of $\ll$ to the boundary. The Budic-Sachs and Szabados approaches introduced modifications in the directions (a) and (c). But for deeper changes, including (b), it is convenient to study first a simplified case, free of bothering identification rules.

3.3 Harris' universal construction

Harris [26, 27] (see also updated [29]) focused on the less problematic future part $M, \hat{M}, \hat{\partial} M$ or *future chronological (chr) boundary*. This is useful from the practical viewpoint: in many cases one may be interested only in what happens "towards the future" (or the past) with no annoying discussions about identifications –and in spacetimes such as the globally hyperbolic ones, this always will happen. But this viewpoint will be also useful to attain a general definition of c-boundary: on one hand, the universal properties of the chronological boundary will give a support for the c-boundary, on the other, the natural topology for the chr boundary, will suggest a natural topology for the c-boundary.

As Harris focuses on $\hat{\partial} M$ as a point set, the first task will be to extend $\ll$ to all $\hat{\partial} M$. He does not try to extend the causal relation $\leq$, and we will not worry about it (we will give some brief comments on extended $\leq$ in Subsection 5.5).

3.3.1 The chronological relation in $\hat{M}$

Let M be a strongly causal spacetime, and $\hat{M}$ be its future GKP (pre)completion. Following GKP approach, Harris defined a chronological relation extended to the completion $\hat{M}$, $\ll^c$, as:

$$\begin{array}{lcl} x \ll^c y & \Leftrightarrow & x \ll y, \\ x \ll^c Q & \Leftrightarrow & x \in Q, \\ P \ll^c y & \Leftrightarrow & P \subset I^-(z) \text{ for some } z \ll y \\ P \ll^c Q & \Leftrightarrow & P \subset I^-(y) \text{ for some } y \in Q \end{array} \tag{3.2}$$

for any $x, y \in M$, $P, Q \in \hat{\partial} M$. As Harris points out, $\ll^c$ becomes nice when M is *past-determined*, i.e.: $x \ll z$ holds if $I^-(x) \subset I^-(y)$ for some $y \ll z$. Otherwise, the following difficulty appears. Consider $x \in M$ and its past $P = I^-(x)$. Because of the different cases in the definition of $\ll^c$, perhaps $x \not\ll^c y$ but, when one removes x from the spacetime, $P \ll^c y$ (Fig. 6).

In order to overcome this difficulty, Harris will also consider a *past-determined chronological relation* $\ll^P$ in $\hat{M}$, which can be reformulated as follows:

$$P \ll^P Q \Leftrightarrow \uparrow P \cap Q \neq \emptyset, \qquad \text{for any IPs } P, Q \in \hat{M}. \tag{3.3}$$

[4]Apart from Kuang and Liang counterexamples, Harris not only pointed out that the problems of the topology of $\partial \mathbb{L}^n$ were inherent to GKP-type constructions (see footnote 3), but also gave an example which suggested that the identifications may yield "incomplete completions", see [26, Appendix].

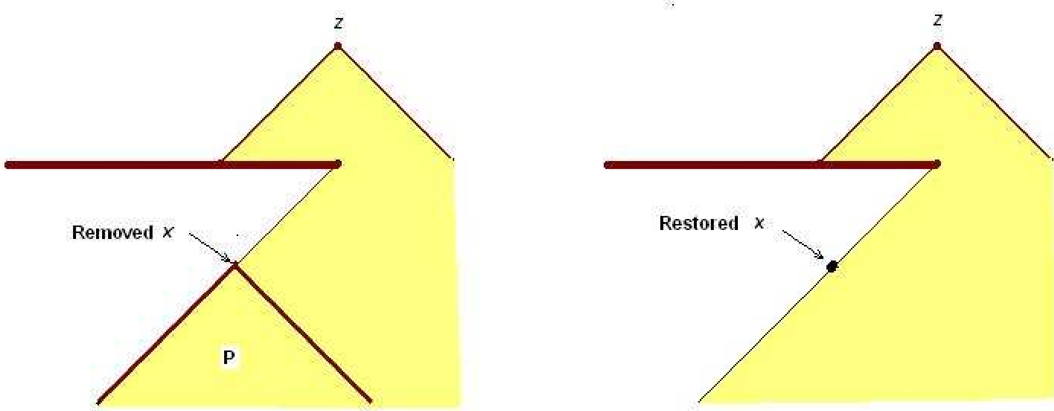

Figure 6: On the right, $Y = \mathbb{L}^2 \backslash \{(0,x) : x \leq 0\}$; on the left, X is equal to Y with the point x removed (X, Y are not past-determined). The natural inclusion $i : X \to Y$ cannot be extended to a chronological map $\hat{\imath} : \hat{X} \to \hat{Y}$ if one considers the chronological relation $\ll^c$ in the completions. In fact $P \ll^c z$ but $x = \hat{\imath}(P) \not\ll^c z$.

Notice that $\ll^P$ considers both, TIPs and PIPs on the same footing. Nevertheless, $\ll^P$ introduces new (spurious) relations in the original spacetime M –so that it will be past-determined. Sometimes, this difficulty will make necessary to focus on the class of past-determined spacetimes (compare with Remark 5.1 below).

3.3.2 Category of chronological sets

Next, one has to construct a category of spaces with includes both, strongly causal spacetimes and spacetimes-with-boundary.

Definition 3.1 *A* chronological set $(X, \ll)$ *is a set X endowed with a binary relation $\ll$ which is transitive, anti-reflexive and satisfies:*

(i) it contains no isolates: each x satisfies $x \ll y$ or $y \ll x$ for some $y \in X$, and

(ii) it is chronologically separable*, that is, there exists a countable $\mathcal{S} \subset X$ which is* chronologically dense*: for all $x \ll y$ there exists $s \in \mathcal{S}$ such that $x \ll s \ll y$.*

All these conditions are satisfied trivially in chronological spacetimes. About the inexistence of isolates, recall that in spacetimes each x satisfies both $x \ll y$ and $y' \ll x$, for some y, y'. Nevertheless, to impose only one of them, becomes natural for points of the boundary. The chronologically dense subset $\mathcal{S}$ will ensure appropriate technical properties.

Notice that we have not defined a topology in $(X, \ll)$ yet. However, definitions for spacetimes on causality and GKP construction ($I^{\pm}(x)$, IPs, TIPs,...) can be translated immediately. Future-directed timelike curves are replaced naturally by *future-directed* (or *chronological*) *chains*: sequences $c = \{x_j\}_j$ such that $x_j \ll x_{j+1}$ for all j. Chronological separability ensures that IPs coincide with pasts of future-directed chains.

Definition 3.2 *A point $x \in X$ is a* future limit *of the future-directed chain c if $I^-(c) = I^-(x)$. X is called* future-complete *if any such future-directed chain has a future limit in X.*

Future-directed chains provide a closer analogy with the Cauchy completion of a metric space, as they play a role similar to Cauchy sequences.

Definition 3.3 *The* chronological boundary $\hat{\partial}X$ *of $(X, \ll)$ is the set of all the TIPs of X, and the* chronological completion $\hat{X}$ *is the union $X \cup \hat{\partial}X$. The latter is also regarded as a chronological space with the extended relation $\ll^c$ in (3.2).*

Due to the generality of chronological spaces, there are two conditions which will be always imposed and are automatically satisfied in both, strongly causal spacetimes and their completions:

- *Past-distinguishing*: $x \neq y \Rightarrow I^-(x) \neq I^-(y)$. This allows to identify each point $x \in X$ with its PIP, $I^-(x)$, and ensures the uniqueness of the future limit for chronological sequences. Notice that strong causality is not considered for $(X, \ll)$: first, a topology is needed for its definition, but even when this is done, a strongly causal spacetime might have, in principle, boundary points where such strong causality were violated [54].

- *Past-regularity*: $I^-(x)$ is an IP for all $x \in X$. This condition is completely natural, but one might be inclined to drop it in same cases. For example, remove the negative time axis of $\mathbb{L}^2$, and consider its chronological completion $\hat{X}$ (see Fig. 4). Each removed point $(t, 0), t \leq 0$ yields naturally two boundary points $(t, 0^-), (t, 0^+)$. Then, identifying $(0, 0^-)$ and $(0, 0^+)$, one obtains a natural *non* past-regular chronological set.

A natural morphism between chronological spaces X, Y will preserve the chronology. But as future limits become fundamental for chronological completions, we will consider morphisms which also preserve these limits:

Definition 3.4 *A function $f : X \to Y$ between chronological sets is* future-continuous *if it:*
(i) is chronological: $x_1 \ll x_2 \Rightarrow f(x_1) \ll f(x_2)$, and
(ii) preserves future limits: if x is the future limit for a future-directed chain $\{x_j\}_j$ then so is $f(x)$ for the (necessarily future) chain $\{f(x_j)\}_j$.

Remark 3.5 If X, Y are strongly causal spacetimes (and, thus, endowed with Alexandrov topology, which comes directly from chronology) a function is future and past continuous iff it is continuous and carries timelike curves to timelike curves [26, p. 5433s]. Nevertheless, future-continuity does not ensure continuity (consider in $\mathbb{L}^2$ the map $(t, x) \mapsto (t-1, x)$ for $t \leq 0$ and equal to the identity otherwise).

Definitions 3.1, 3.4 allow to consider the category $\mathcal{C}$ with objects the past-distinguishing past-regular chronological sets and morphism the future-continuous functions. As the morphisms preserve completeness, we can also consider the subcategory $\mathcal{C}_0$ whose objects are all the future-complete $\mathcal{C}$-objects. The next step in Harris approach is to show that every morphism $f : X \to Y$ extends to a morphism $\hat{f} : \hat{X} \to \hat{Y}$ in order to arrive at a functor $\hat{}: \mathcal{C} \to \mathcal{C}_0$. The only possibility for the definition of $\hat{f}$ is obvious: any point P of $\hat{X}$ can be regarded as an IP and, thus, represented as $I^-(c)$ for some future-directed chain c, so, $\hat{f}(P)$ is defined as $I^-[f(c)] \in \hat{Y}$. Unfortunately, $\hat{f}$

does not necessarily preserve the chronology (Fig. 6). So, one is forced to consider the subcategories $\mathcal{C}^{pd}$ and $\mathcal{C}_0^{pd}$ (of $\mathcal{C}$ and $\mathcal{C}_0$, resp.) whose objects are past determined chronological sets. The following categorically universal result can be then obtained [26]:

Theorem 3.6 *In the subcategory of past-determined chronological sets $\mathcal{C}^{pd}$: (i) each morphism $f : X \to Y$ extends naturally and univocally to a unique morphism $\hat{f} : \hat{X} \to \hat{Y} \in \mathcal{C}_0^{pd}$, (ii) future-completion $\widehat{\ }: \mathcal{C}^{pd} \to \mathcal{C}_0^{pd}$ is a functor, and (iii) the standard future injection is a natural transformation between functors.*

The drawback of this universality is that it applies only in the past-determined category. Nevertheless, Harris also defined a natural functor **pd**: $\mathcal{C} \to \mathcal{C}^{pd}$ of past-determination. A further study of this functor shows the naturality of the GKP construction as a minimal way of "future-completing" any past-regular, past-distinguishing chronological set.

3.3.3 Chronological topology

In order to define a topology in any $(X, \ll)$ associated to the chronological relation, some first options must be disregarded:

Remark 3.7 (a) As we have already commented in Subsection 3.1, general examples show that plain Alexandrov's topology yields unsatisfactory properties in simple cases. A different type of example is cited by Harris [27, Figure 2] in the context of future boundaries. This may be illuminating and will be revisited below (Fig. 10). Consider the chronological space X obtained by removing the negative spacelike semi-axis $x \leq 0$ from $\mathbb{L}^2$. Each removed point $(t = 0, x)$ yields naturally a boundary point $P_x (= I^-(0, x)) \in \hat{\partial} X$. In the completion $\hat{X}$, the past of $(1, 0)$ would yield an Alexandrov open subset which contains P_0 but no other P_x. So, $P_{1/n} \not\to P_0$. But, as we are looking only at past sets, the convergence of the sequence seems desirable.

(b) The GKP topology would not be appropriate, even though now the problem of the identifications between future and past preboundary points does not appear. Among the reasons discussed previously, the anomalous topology for $\hat{\partial}\mathbb{L}^n$ (footnote 3) is especially unfavorable now, as $\mathbb{L}^n$ is globally hyperbolic and, thus, there are no identifications between future and past boundary points.

Instead, Harris defined a *limit operator* $\hat{L}$: for any sequence $\sigma = \{x_n\}_n \subset X$,

$$x \in \hat{L}(\sigma) \Leftrightarrow \left\{ \begin{array}{l} y \ll x \Rightarrow y \ll x_n \\ I^-(x) \subsetneq P(\in \hat{M}) \Rightarrow \exists z \in P : z \not\ll x_n \end{array} \right. \quad \text{for large } n$$

or, equivalently, in Flores' reformulation:

$$x \in \hat{L}(\sigma) \Leftrightarrow \left\{ \begin{array}{l} I^-(x) \subset LI\{I^-(x_n)\} \\ I^-(x) \,\text{is a maximal IP in } LS\{I^-(x_n)\} \end{array} \right. \tag{3.4}$$

where LS, LI denote the lim-sup and lim-inf operators in set theory, that is, for any sequence of subsets $A_n \subseteq X$, $LS(\{A_n\})$ (resp. $LI(\{A_n\})$) contains any $x \in X$ which belongs to infinitely many A_n (resp. all A_n for n sufficiently large). This limit operator defines the closed subsets for a topology, in fact:

Definition 3.8 *Let X be a past-regular chronological set. The* future-chronological topology *is the one such that: $C \subseteq X$ is closed if and only if for any sequence σ in C, necessarily $\hat{L}(\sigma) \subseteq C$.*

Notice: (i) as X is chronologically separable, the so-obtained topology is second countable (any second countable topology can be characterized by such a limit operator), (ii) points are closed because of past regularity, i.e., the future-chronological topology is T_1.

The following properties suggest that future-chronological topology has been chosen properly[5]:

1. For any future-directed chain $\sigma = \{x_n\}$: $x \in \hat{L}(\sigma) \Leftrightarrow \{x_n\} \to x$.
2. The standard injection $X \hookrightarrow \hat{X}$ is a homeomorphism into its image, and X is dense in $\hat{X}$.
3. If X is a strongly causal spacetime, X has the manifold topology and $\hat{\partial}X$ is closed in $\hat{X}$.
4. For $X = \mathbb{L}^n$, the chr-boundary is the usual cone with its natural topology.

Additionally, the topology satisfies an interesting property of quasi-compactness [14, Sect. 5]: any sequence $\{x_n\}$ with $LS(\{x_n\}) \neq \emptyset$ admits a subsequence with a non-empty limit.

Nevertheless, in spite of these nice properties, there is an undesirable property. As we have already commented (Remark 3.7(a)), sets type $I^{\pm}(x)$ are not always open now. This may be natural when only convergence of IP's is being considered (forgetting what happens for IF's). Nevertheless, this also implies that, in the future chronology, points of the boundary are not T_2 related to points of the spacetime (Fig. 10).

Recall also that there is no a general categorical result for the future-chronological topology. The reason is the following. Consider a future-continuous function $f : X \to Y$ (with X, Y past-determined) and its extension to the completions $\hat{f} : \hat{X} \to \hat{Y}$. If X, Y are endowed with the chronological topology, *the continuity of f is not sufficient to ensure the continuity of $\hat{f}$*. This happens so easily (Fig. 7) that seems to be an inherent obstruction to the categorical approach: if the objects in the categories $\mathcal{C}, \mathcal{C}_0$ are endowed with the chr-topology, and the morphisms are assumed to be also continuous, then future completion does not yield a functorial relation between $\mathcal{C}$ and $\mathcal{C}_0$ (or subcategories such as $\mathcal{C}^{pd}, \mathcal{C}_0^{pd}$). Moreover, it is hard to think that some natural definition of the topology of the c-boundary might satisfy such a universal property.

At any case, Harris also proved that, in the subcategory of *chronological sets with spacelike boundaries* (see Subsection 5.5), such a functorial relation is still possible, and the universality of chr-boundary in the sense of categories, holds. (Notice that in the case of spacelike boundaries, the problem of identifications between future and past boundary points do not appear.) Again, this yields a strong support for chr-topology.

3.4 Summary

We retain the following elements of previous approaches:

1. The GKP precompletion of $M^\sharp$ (spacetime + preboundary points) is natural, but its chronology, topology and identifications (either as in the GKP approach or in related methods) become suspicious.

[5] This topology depends on $\ll$ and, so, would depend on the choice $\ll^c$ in (3.2) for a completed spacetime. Nevertheless, Harris emphasizes [29] that topological properties will be independent of past determination –and, then, on some subtleties of this choice.

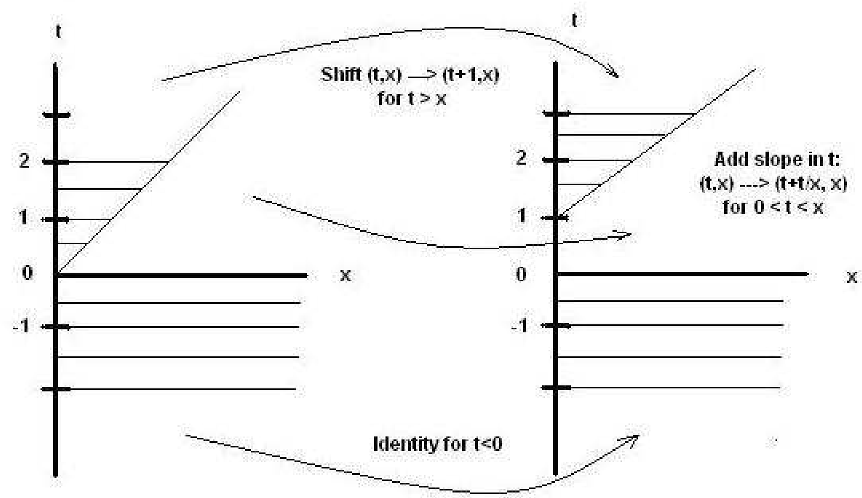

Figure 7: Bijective chronological map of $\mathbb{L}^1 \times (0, \infty)(\subset \mathbb{L}^2)$ in itself. It is continuous with the chr-topology (and future-continuous) but cannot be extended continuously to the completions, as the origin "expands" to the vertical segment $[0, 1] \times \{0\}$ in the boundary, [27, p. 569].

2. For the future boundary $\hat{\partial}M$, Harris universal properties for $(M, \ll)$ as a chronological set holds. Even though there is some restriction (class of past-determined chronological sets), perhaps some different extension of the relation $\ll$ to the boundary might drop it. At any case, the GKP construction of $\hat{M}$ finds a good support as a point set and a chronological set.

3. Harris limit operator $\hat{L}$ for the topology of $\hat{M}$ (avoiding GKP/Alexandrov topologies) becomes natural as a convergence of past sets. Nevertheless, as the future parts are not taken into account, points of $\hat{\partial}M$ and M may be not T_2 related. Its universal properties are restricted to the class of spacetimes with spacelike boundaries. However, this seems inherent to the conformal character of the construction. Examples suggest that this restriction would be unavoidable for the universal properties of the topology, in any notion of the c-boundary.

4. At any case, it is necessary to relate $\hat{\partial}M$ and $\check{\partial}M$. Even though the procedure was not clear, the Szabados relation, which refined Budic and Sachs', became promising.

5. Additionally, Harris [28] introduced some tools to compute the boundary in standard static spacetimes (refined later in [14, 29]) and quotient spacetimes.

And, of course, one also had a set of worrying counterexamples, as those by Kuang and Liang!

4 Intermission: holography and waves come into play

4.1 AdS/CFT correspondence and boundaries

Very roughly, the idea of holography is that physics in a region is encoded by some fundamental degrees of freedom in the boundary of this region –the "hologram" of the original region. The seminal idea appeared by studying the entropy of black holes, which is proportional to the area of the event horizon –a surprising property, as entropy is an "extensive" magnitude and one would expect proportionality to the volume. So, G. 't Hooft [32] and L. Susskind [53] suggested that the nature of quantum gravity might be holographic. The most rigorous realization of the holographic principle is Madacena's AdS/CFT correspondence. This is a conjectured equivalence between: (i) string theory on a space (typically, $\mathrm{AdS}_5 \times \mathbb{S}^5$, or the product of anti de-Sitter by other compact manifold), and (ii) a Quantum Field Theory (say, a Conformal Field Theory) on the *conformal boundary* of this space, which behaves as a hologram of inferior dimension.

We are interested in this holographic boundary. It is not difficult to check that $\mathrm{AdS}_5 \times \mathbb{S}^5$ is conformally equivalent to $(\mathbb{R}^6 \setminus \{0\}) \times \mathbb{L}^4$ and, thus, to $\mathbb{L}^{10} \setminus \mathbb{L}^4$ (a timelike linear subspace $\mathbb{L}^4$ is removed from $\mathbb{L}^{10}$). In fact, taking into account the Poincaré representation of AdS_5, $g_P = (dy^2 + g_{\mathbb{L}^4})/y^2, y > 0$ the metric $g = g_P + g_{\mathbb{S}^5}$ in $\mathrm{AdS}_5 \times \mathbb{S}^5$ is conformally equivalent to:

$$y^2 g = dy^2 + g_{\mathbb{L}^4} + y^2 g_{\mathbb{S}^5} \equiv g_{\mathbb{R}^6} + g_{\mathbb{L}^4} \qquad \text{for } \ y > 0.$$

The conformal boundary obtained by means of the inclusion $(\mathrm{AdS}_5 \times \mathbb{S}^5 \approx) \mathbb{R}^6 \setminus \{0\} \times \mathbb{L}^4 \hookrightarrow \mathbb{L}^{10}$, restores the removed subspace $\mathbb{L}^4$. This is also the expected causal boundary and, so, it is not relevant in this case which one of the two boundaries is chosen.

4.2 Holography on plane waves

Nevertheless, the situation will change when one consider holography on plane waves. A priori, such a holography is interesting because: (a) some plane waves provide exact backgrounds for string theory (as all curvature invariants vanish), and (b) every spacetime has a plane wave as a limit (Penrose limit [45]). But the string community was not truly interested in this holography until Berenstein, Maldacena and Nastase detailed correspondence [4] between 10d-string theory on plane waves and 4d-super Yang-Mills theory. Moreover, Blau, Figueroa-O'Farrill, Hull, Papadopoulos (BFHP) took a lightlike geodesic in $\mathrm{AdS}_5 \times \mathbb{S}^5$ which rotates on $\mathbb{S}^5$, considered its Penrose limit and identified its dual in field theory [8].

Berenstein and Nastase [5] revised previous approach. They found that, unexpectedly, the conformal boundary for BFHP wave is 1-dimensional. In their conformal embedding, the boundary was a null line with a rather surprising role of "future and past" (Fig. 8). This additional reduction of the dimension became relevant, as it opened new possibilities for the holography (the holographic dual can be described by a quantum mechanical system –a matrix model).

4.3 Is conformal boundary the right choice?

Marolf and Ross [40] realized that the usage of conformal boundary was limited to very particular plane waves –say, the conformally flat ones. As the causal boundary is conformally invariant and applicable to many more waves, they proposed to study such a boundary. Due to the results for

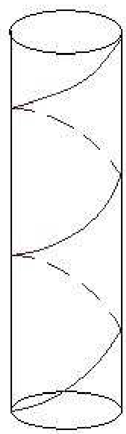

Figure 8: The depicted "helix" in Ein_2 represents the 1-dimensional boundary of BFHP wave. One can arrive at this lightlike boundary by means of both, future and past directed timelike curves.

the BFHP wave, the problem of the identifications became essential and, so, they introduced their own proposal, which will be discussed below. Then, they proved that the 1-dimensional character of the conformal boundary, is reproduced for the causal boundary not only for BFHP plane wave but also for other plane waves.

Even though they made a concrete choice of c-boundary, their results held by assuming a minimal requirement, namely: for any sequence $\{p_n\}_n \subset M$, if the PIP's $I^-(p_n)$ approach (in any reasonable sense) some TIP P and the PIF's $I^+(p_n)$ approach some TIF F then P, F represent the same point. This puts forward the problem of the identifications.

As we will see next, Marolf and Ross study of c-boundary was widely extended by Flores [13], and the systematic computation of the c-boundaries of wave type spacetimes was carried out by Flores and Sánchez [17]. But, prior to this, the following conclusion is clear:

> In the AdS/CFT correspondence, the boundary of the spacetime was used in an elementary way by means of conformal embeddings. But when plane wave backgrounds come into play, this is no longer satisfactory: the causal boundary must be used and the problem of identifications becomes essential.

5 Reconstructing the c-boundary

5.1 Marolf-Ross seminal idea

As pointed out in the Introduction, the key role of Marolf-Ross approach [41] is to consider the c-boundary $\partial_{MR}M$ as the set of all the pairs S-related (see (3.1). That is, for any strongly causal

spacetime M (or even any distinguishing and regular chronological set)

$$\begin{aligned} \partial_{MR}M = \quad & \{(P,F) : P \text{ is a TIP}, \ F \text{ is a TIF}, \text{and } P \sim_S F\} \\ & \cup\{(P,\emptyset) : P \text{ is a TIP}, \text{and } P \not\sim_S F \text{ for any TIF } F\} \\ & \cup\{(\emptyset,F) : F \text{ is a TIF}, \text{and } P \not\sim_S F \text{ for any TIP } P\} \end{aligned} \tag{5.1}$$

In particular, if $P \sim_S F \sim_S P'$ then $(P,F),(P',F)$ are regarded as two different boundary points. Among the advantages of this viewpoint, one is apparent: the chronological relation can be extended in an obvious way to the boundary (Fig. 9):

$$(P,F)\overline{\ll}(P',F') \Leftrightarrow F \cap P' \neq \emptyset, \quad \forall (P,F),(P',F') \in \partial_{MR}M \tag{5.2}$$

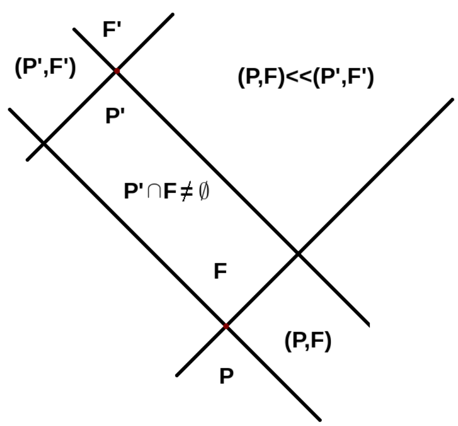

Figure 9: When the points of the completion are regarded as pairs, the chronological relation emerges naturally from (5.2).

This simple definition is satisfactory, as one can prove that $\overline{\ll}$ satisfies: (i) is transitive, and (ii) does not introduces additional (spurious) relations in M –when applied to PIP's and PIF's[6].

Remark 5.1 The essential uniqueness of the extended chronological relation for pairs contrasts with previous approaches. For example, Harris' choice $\ll^c$ (see (3.2)) became standard and natural, but we have remarked the problems concerning past-determination. Taking into account (5.2), a subtler extension $\ll^e$ would seem also reasonable now: $P \ll^e P'$ iff P' is intersected by some IF, F, included and maximal in $\uparrow P$, such that $P \sim_S F$. Apparently, this is an extended chronological relation in $\hat{M}$ which does not introduce spurious relations when P, P' are TIPs.

Moreover, other possibilities of the chronological extension (applicable to completions of chronological sets or spacetimes) might appear. For example, most of the striking differences between the abstract chronological relations in $(X,\ll)$ and the relation $\ll$ in a spacetime M, comes from the fact that the latter is defined by means of paths. As any chronological space can be also endowed

[6]Notice that such a $\overline{\ll}$ was essentially introduced by Szabados [54] but, under his approach, conditions (i) and (ii) could not be fulfilled: the problem was inherent in the fact that all the elements connected by $\sim_S$ represented the same boundary point, (see [41] or Example 5.7 below).

with a topology, one can generate a new chronological relation which collects the idea of paths, either directly (in spacetimes) or by mean of chains (in chronological sets). So, start with some chronological set $(X, \ll)$ (eventually, take $X = \hat{X}, \ll = \ll^c$) and define: two points $x, y \in X$ are *path-chronologically related*, $x \ll_p y$ if for every open covering $\{U_\alpha\}_\alpha$ of X there exists a chain $x = x_0 \ll x_1 \ll \ldots \ll x_k = y$ such that each consecutive pair x_i, x_{i+1} is contained in some U_α. Any spacetime is *path-chronological* i.e. $\ll = \ll_p$; if the extended relation $\ll^c$ in a topologized completion were not, one could explore to change $\ll^c$ by $\ll_p^c$.

However such alternatives, which may be reasonable under previous approaches, cannot be regarded as serious alternatives to (5.2) for pairs.

So, Marolf-Ross' ideas introduce a new and exciting viewpoint, which was complemented by suggesting two possible topologies for the boundary. But, in order to understand the exact role of MR completion, let us consider first the general notion of completion introduced by Flores [13].

5.2 Flores' general completions: point set and chronological extension

Let us retain the most general elements of Harris' setting, extending when necessary past elements to future ones. That is, consider, among other straightforward definitions, a *chronological set* $(X, \ll)$, (future or past directed) *chains* and (past and future) *limit operators* $\hat{L}, \check{L}$ for any sequence $\sigma \subset X$ as in (3.4). For any future-directed chain $\eta = \{x_n\}_n$ the inclusions $I^-(x_n) \subset I^-(x_{n+1})$ and $I^+(x_n) \supset I^+(x_{n+1})$ yield directly that the lim sup LS and lim inf LI of each sequence $I^\pm(x_n)\}_n$ coincide and, then:

Proposition 5.2 *Let $(X, \ll)$ be a (past and future) regular chronological set. For any future (resp. past) chain η:*

$$x \in \hat{L}(\eta) \Leftrightarrow I^-(x) = I^-(\eta), \qquad x \in \check{L}(\eta) \Leftrightarrow I^+(x) \subset \uparrow \eta \text{ and is maximal in } \uparrow \eta$$
$$(\text{resp. } x \in \check{L}(\eta) \Leftrightarrow I^+(x) = I^+(\eta), \qquad x \in \hat{L}(\eta) \Leftrightarrow I^-(x) \subset \downarrow \eta \text{ and is maximal in } \downarrow \eta).$$

The natural notion of completion for $(X, \ll)$ must comprise that any chain have an endpoint. So, we need two previous concepts: the first one is the set-point space where this completion makes sense, the second is the notion of *endpoint of a chain*.

Let X_p, X_f be, resp., the sets of all the past sets ($P \subset X$, $P = I^-(P)$) and future sets ($F \subset X$, $F = I^+(F)$) –not necessarily IPs or IFs. Assuming that the chronological set is *weakly distinguishing* (i.e., either future or past distinguishing) the original chronological set injects naturally,

$$\mathbf{i} : X \to X_p \times X_f, \qquad x \rightarrowtail (I^-(x), I^+(x)).$$

So, as a first conclusion, any completion $\bar{X}$ would satisfy $\mathbf{i}(X) \subset \bar{X} \subset X_p \times X_f$ as a point set.

The notion of endpoint is subtler. Recall that we have not defined a topology yet, even though the operator $\hat{L}$ does define a natural topology in $\hat{X}$. In principle, an endpoint is not the same thing that a *limit*, but Proposition 5.2 is a clear guide. The essence of an endpoint (P, F) for, say, a future-directed chain η deals with future convergence, and naturally one must impose $P = I^-(\eta)$. Nevertheless, some control is necessary for the F component, and this is carried out by means of the operator $\check{L}$. As this operator yields IFs, let us introduce first a *decomposition operator*, dec. By using Zorn's lemma, it is easy to show that any $P \in X_p$ can be written as an union $P = \cup_\alpha P_\alpha$,

where the set $\{P_\alpha : \alpha \in I\}$ contains all the IP's included in P which are *maximal* under the relation of inclusion. Consider the following operator, which applies on past sets and, dually, on future sets:

$$\mathrm{dec}(P) = \{P_\alpha : P_\alpha \text{ is a maximal IP included in } P, \forall \alpha \in I\}, \qquad \text{for any } P \in X_p. \tag{5.3}$$

Now, we have the elements for the definitions.

Definition 5.3 *Let X be a weakly distinguishing chronological set X.*

(a) A pair $(P, F) \in X_p \times X_f$ is an endpoint *of a future (resp. past) chain $\eta \subset X$ if*

$$P = I^-(\eta),\ dec(F) \subset \check{L}(\eta) \qquad (resp.\ \ F = I^+(\eta),\ dec(P) \subset \hat{L}(\eta)).$$

(b) X is complete *if any chain η in X has some endpoint in X.*

(c) A set $\bar{X}, \mathbf{i}(X) \subset \bar{X} \subset X_p \times X_f$ is a completion *of X if any chain η in X has some endpoint in $\bar{X}$. In this case, the* extended chronological relation $\overline{\ll}$ *on $\bar{X}$ is defined as in (5.2).*

The boundary of the completion is then $\partial X := \bar{X} \backslash \mathbf{i}(X)$. Some natural properties show the consistency of the definitions (see [13] for detailed proofs): (i) $(\bar{X}, \overline{\ll})$ becomes a weakly distinguishing chronological set, (ii) the injection $\mathbf{i}$ is a chronological map between $(X, \ll)$ and $(\bar{X}, \overline{\ll})$, (iii) the image $\mathbf{i}(X)$ is chronologically dense in $\bar{X}$, (iv) no spurious chronological relations are introduced by $\overline{\ll}$ in $\mathbf{i}(X)$, and (v) any completion $(\bar{X}, \overline{\ll})$ of X is a complete chronological set. We emphasize the natural role of $\overline{\ll}$ for these properties. For example, by using a different extended chronology, Harris had constructed a spacetime, which suggested a generic counterexample to property (v) –see footnote 4.

5.3 Minimal completions and Marolf-Ross one

Previous notion of completion is very general and includes not only Marolf-Ross one but, for example, GKP precompletion (looking each IP P as the pair $(P, \emptyset)$ and each IF F as $(\emptyset, F)$). In order to have more efficient completions, we must restrict to appropriate *minimal* ones.

Consider the set $\mathcal{C}_X$ of all the completions of a weakly distinguishing X. Now, delete from $\mathcal{C}_X$ those completions which are still a completion when some of the points of the boundary is removed. The resulting subset $\mathcal{C}_X^*$, is not empty (for example, GKP precompletion remains there), and a partial ordering $\leq$ is defined as follows. Let $\bar{X}^I, \bar{X}^J \in \mathcal{C}_X^*$, we put

$$\bar{X}^I \leq \bar{X}^J$$

if for each $(P_i, F_i) \in \partial X^I$ there exists some $S_i \subset \partial X^J$ such that the following properties are fulfilled:

(a) The set of all the S_i's is a partition of ∂X^J.

(b) If some chain η in X has an endpoint in S_i then (P_i, F_i) is also an endpoint of η.

(c) If some S_i contains only a point $S_i = \{(P, F)\}$ then $\mathrm{dec}(P_i) \subset \mathrm{dec}(P)$ and $\mathrm{dec}(F_i) \subset \mathrm{dec}(F)$.

Recall that (a) implies the existence of an onto map $\Pi : \partial X^J \to \partial X^I$. By (b) if all the boundary points in S_i were replaced by $\Pi(S_i) = (P_i, F_i)$, then one would still obtain a completion (however, recall that S_i has at most two points). Finally, (c) ensures that, if (P, F) is replaced by (P_i, F_i) then the latter is simpler than the former (and the chains will have still endpoints). The existence of minimal elements for the partial order $\leq$ is guaranteed by using Zorn's lemma.

Such minimal completions are also called *chronological completions* in [13]. They fulfill very satisfactory properties for strongly causal spacetimes (see [13, Th. 7.4]):

Theorem 5.4 *Let M be a strongly causal spacetime. A completion $\bar{M} \subset M_p \times M_f$ is minimal if and only if its boundary ∂M satisfies the following properties:*

(i) Every TIP and TIF in M is the component of some pair in ∂M.

(ii) If $(P, F) \in \partial M$ and $P \neq \emptyset \neq F$, then P is a TIP, F is a TIF, and $P \sim_S F$.

(iii) If $(P, F) \in \partial M$ and $F = \emptyset$ (resp. $P = \emptyset$) then P (resp. F) is a (non-empty) terminal indecomposable set and is not S-related to any other set.

(iv) If $(P, F_1), (P, F_2) \in \partial M$ and $F_1 \neq F_2$ (resp., $(P_1, F), (P_2, F) \in \partial M$ and $P_1 \neq P_2$) then F_i (resp. P_i), $i = 1, 2$, does not appear in another pair of ∂M.

Moreover, Marolf-Ross completion (5.1) is the maximum completion (the biggest as a subset of $M_p \times M_f$) which satisfies properties (i), (ii) and (iii).

Notice that property (i) is necessary to make $\bar{M}$ a completion, and (iv) to make it minimal. The appearance of properties (ii) and (iii) become specially relevant. First, *Szabados relation appears naturally, it is not imposed a priori as in previous approaches.* And, second, as a straightforward consequence of Marolf-Ross' definitions:

Corollary 5.5 *Let M be a strongly causal spacetime, $\bar{M}$ a minimal completion and $\bar{M}_{MR}$ Marolf-Ross one. Then $\bar{M} \subset \bar{M}_{MR}$, and the inclusion is a chronological map*[7]*.*

Remark 5.6 (a) The appearance of S-relation $\sim_S$ for all the points in the completion (Theorem 5.4), is a more general fact for (say, regular, distinguishing) chronological sets. Nevertheless, there is an important reason for the restriction to the class of strongly causal spacetimes. According to [54, Proposition 5.1]: *(i) A spacetime M is strongly causal iff $I^+(p) \sim_S I^-(p)$ for all $p \in M$, and (ii) if M is strongly causal then $I^-(p) \sim_S F$ iff $F = I^+(p)$* (and analogously for $I^+(p)$). That is, when M is not strongly causal, then $(I^-(p), I^+(p))$ is not always a pair taken into account in Marolf-Ross completion (such pairs should be added directly in order to get a completion), and pairs type $(P, I^+(p))$, $(I^-(p), F)$ with $p \in M$ and P, F terminal sets, may appear.

(b) For a strongly causal spacetime M, one would be tempted to consider $\bar{M}_{MR}$ as the smallest completion which includes all the minimal ones. Unfortunately, this property do not hold: even when there exist only one minimal completion, $\bar{M}_{min}$, Marolf-Ross one may be strictly bigger than $\bar{M}_{min}$ (see Example 5.7(2) below).

(c) However, it is clear that properties (i), (ii) (iii), which appeared axiomatically in previous approaches, now receive a strong support. So, call any completion which fulfills the three properties an *admissible completion*[8]. Then, Marolf-Ross completion is singled out as the maximum admissible completion of M.

[7] In fact, it is also future (and past) continuous (Defn. 3.4), and a chronological isomorphism onto its image.

[8] Such a completion, as well as some results below, will be developed in [16].

The following examples, taken essentially from [13, Example 9], [41, Appendix], stress the role of Marolf-Ross completion as a non-minimal one, as well as the importance of pairings for the consistency of $\overline{\ll}$.

Example 5.7 (1) Delete from $\mathbb{L}^3$ the coordinate semi-planes: $XT^+ = \{(t, x, y = 0) : 0 \leq t\}$ and $YT^- = \{(t, x = 0, y) : t \leq 0\}$. Clearly, there are four terminal sets associated with the removed origin: two TIPs P_1, P_2 (corresponding to the $x > 0$ and $x < 0$ directions, resp.) and two TIFs F_1, F_2 (corresponding to $y > 0$ and $y < 0$, resp.); moreover $P_i \sim_S F_j$ for $i, j = 1, 2$. Marolf-Ross completion includes then the four pairs (P_i, F_j) as boundary points. Nevertheless, minimal completions are obtained by attaching only $(P_1, F_1), (P_2, F_2)$ or only $(P_1, F_2), (P_2, F_1)$. Both are naturally included in Marolf-Ross. In this concrete example, the two minimal completions are isomorphic in a natural sense, but this is not expected in general.

(2) Remove in previous example $X^+Y^+ = \{(t = 0, x, y) : 0 \leq x, 0 \leq y\}$. Now, $P_1 \not\sim F_1$ and Marolf-Ross retains the other three pairs. Nevertheless, $(P_1, F_2), (P_2, F_1)$ yield a minimal completion, which is *unique and strictly smaller* than Marolf-Ross one.

(3) Modify slightly example (1) by removing also $Y^+T^+ = \{(t, x = 0, y) : 0 \leq y, 0 \leq t\}$. Now, F_1 splits naturally in two: F_1' which contains $p = (2, 1, 1,)$ and F_1'' (thus $(2, -1, 1) \in F_2''$). The past set P_2, which contains $-p$, is not directly S-related with F_1' according to (3.1). In fact, no point with $t, x, y < 0$ is chronologically related to any with $t, x, y > 0$. None of the pairs would be chronologically related according to $\overline{\ll}$. Nevertheless, according to Szabados identifications, all the terminal sets P_1, P_2, F_2, F_1', F_1'' would collapse to a single boundary point Q. Therefore, according to Szabados, one would have $-p\overline{\ll}Q\overline{\ll}p$. So, either a spurious relation would be introduced in the spacetime, or the transitivity of $\overline{\ll}$ would be violated. No completion in the sense of Defn. 5.3 have such an important drawback.

5.4 The topology

Up to now, the development of the c-boundary has depended on three basic ideas: (a) completions as subsets of $X_p \times X_f$, (b) the extended chronological relation (5.2), and (c) Defn. 5.3 of endpoint. The first one was just a general setting for completions, which includes any previous one. The second one, even though non-trivial, is the natural and apparently unavoidable choice (transitive and free of spurious relations) inside this setting. Only the third one could be thought as a "choice" among some conceivable alternatives. Nevertheless, Defn. 5.3 allows one to have endpoints in such a general way, that any alternative definition would add additional requirements in order to have an endpoint of a chain –that is, the corresponding completion would be a particular case of those already defined. The other two relevant definitions –minimal and Marolf-Ross completions– are selected just by its desirable properties and uniqueness.

For the definition of the topology there exists, in principle, a bigger arbitrariness of the choice. However, we will see that Flores' choice has no reasonable alternative from three viewpoints: (A) the a priori natural choice in the setting, (B) the good mathematical properties obtained a posteriori, and (C) the uniqueness properties. But, prior going further, let us start with an example of topological choice.

This example was already introduced in Remark 3.7(a) (see Fig. 10). Here, one has, for instance, $P \in \hat{L}(\{P_{-1/n}\})$, as we focus only on the future boundary $\hat{\partial}M$. But this also suggests that $Q = (P, F) \in \partial M$ must lie in the limit of the sequences $\{x_n\}, \{Q_n\}$. Marolf and Ross [41]

gave two topologies. For the preferred one (to be described below), $\{x_n\}, \{Q_n\} \not\to Q$, but this was regarded as a non-desirable property. So, they introduced an alternative coarser topology and, then, both sequences converge[9]. Flores [13, Example 10.4 and p. 631] claims that such convergences are not natural, as the intuition of convergence does not take into account what is happening with the future parts of the pairs. Now, recall that there exists a powerful reason to support this last opinion: in the completion $I^-(z)(\equiv I^-(z,\bar{M}))$ contains Q and none of the points in the sequences. So, if $I^{\pm}(z)$ must be open (and this is truly a natural requirement for the topology!) then the sequences cannot converge.

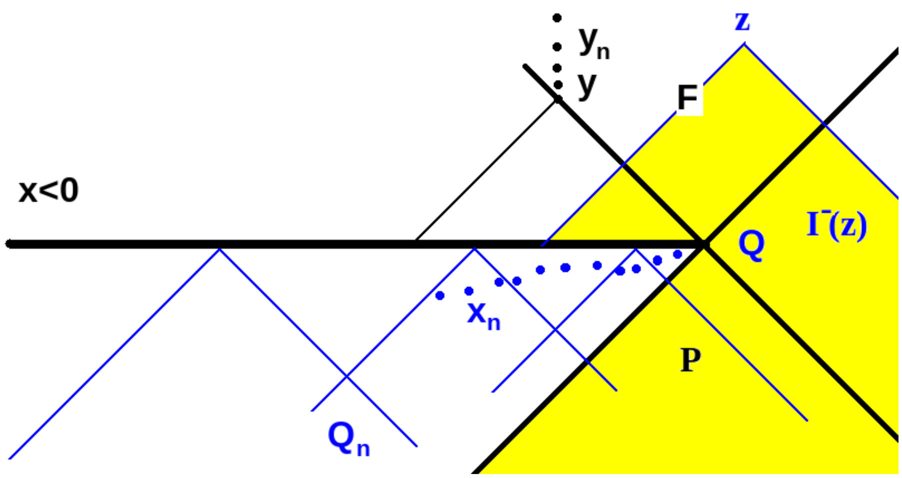

Figure 10: In $\mathbb{L}^2\backslash\{x \leq 0\}$ each point of the removed semi-axis yields two boundary points $(P_x, \emptyset), (\emptyset, F_x)$, when $x < 0$, and a unique point $Q = (P, F)$ for $x = 0$. This spacetime M exemplifies remarkable properties: (a) $I^-(y)$ and P lie in $\hat{L}(\{y_n\}_n)$, that is, $y \in M$ and $P \in \hat{\partial}M$ are not T_2 related for the future chronological topology, (b) if $I^-(z)$ is open in $\bar{M}$, neither the sequence $\{Q_n = (P_{-1/n}, \emptyset) \subset \partial M\}_n$ nor the depicted one $\{x_n\}_n \subset M$ can converge to Q, (c) the lack of global hyperbolicity of M appears just because the boundary is timelike at Q.

Now, we describe Flores' choice. Along the discussion we will focus on $(X, \ll)$ which are both *regular and distinguishing*, even though some items do not require these (mild) conditions.

(A) The natural notion of convergence and topology. Recalling Defn. 5.3 and Prop. 5.2, x is an endpoint of a (past or future-directed) chain σ iff

$$\mathrm{dec}(I^-(x)) \subset \hat{L}(\sigma), \qquad \mathrm{dec}(I^+(x)) \subset \check{L}(\sigma). \tag{5.4}$$

A minimum requirement for the topology is that the endpoint x will be also the limit of the chain σ. So, the simplest natural choice which fulfills this requirement is, obviously:

[9]Really, Marolf and Ross [41, Figure 5] considered a more sophisticated example which comes from [24]. Nevertheless, the intuition for the convergence of the sequences is similar to the simplified example above -and $I^{\pm}(z)$ is not open for some points of their alternative topology.

Definition 5.8 *Let $(X, \ll)$ be regular and distinguishing.*

(1) Given a sequence σ in X, we say that x is a limit of σ, or belongs to the limit operator L on σ *(denoted $x \in L(\sigma)$), if both inclusions (5.4) hold.*

(2) A subset $C \subset X$ is closed *if it contains all the limits of all the sequences contained in C. The* chronological topology *on $(X, \ll)$ is the one generated by the closed sets.*

Of course, this topology can be defined in any chronological set, as in [13]. Perhaps, some alternative option would be conceivable if X were not regular, as in this case endpoint and limit do not coincide for chains. But in the regular case, Defn. 5.8 implies directly that both concepts coincide, and any alternative would be more complicated.

(B) Good properties of the topology. The properties fulfilled by the chronological topology were studied systematically in [13]. Some of them are valid for any completion, but, as one could expect, the full desirable properties hold only for the minimal completions of strongly causal spacetimes or, with some more generality, for their admissible completions (see Remark 5.6(c)). Summing up:

1.- For any completion $\bar{X}$: (i) $\mathbf{i} : X \to \bar{X}$ is a topological imbedding, and $\mathbf{i}(X)$ is dense in $\bar{X}$, (ii) any chain in a complete chronological space (in particular, in any completion) has a limit, and (iii) for a strongly causal spacetime M, the topology of the spacetime coincides with the chronological topology, and any inextensible timelike curve in M has a limit in ∂M.

2.- For any admissible completion $\bar{M}$ of a strongly causal spacetime M, including minimal and Marolf-Ross ones: (i) the boundary ∂M is a closed subset of $\bar{M}$, (ii) $\bar{M}$ is T_1, (iii) if two points $Q, R \in \bar{M}$ are non-Hausdorff related then both lie in ∂M, and (iv) $I^{\pm}(Q)$ is open for any $Q \in \bar{M}$ (for this last property, see also the forthcoming study [16])[10].

(C) Uniqueness properties. The admissible alternatives to the topological chronology for $(X, \ll)$ will be studied in detail in [16]. Here, we announce only that any admissible topology must satisfy both, compatibility with $\ll$ (concretely, $I^{\pm}(x)$ must be open and the endpoints of chains must be also limits of the chains) and compatibility with the set-point limit operators (consider a converging sequence $\{Q_n\}_n \to Q$, looked in $X_p \times X_f$ i.e. $\{(P_n, F_n)\}_n \to (P, F)$; if it happened $P \subset P' \subset LI(P_n), F \subset F' \subset LI(F_n)$, then any neighborhood of (P, F) should contain (P', F')). Among such topologies, the chronological one is selected as the one with best properties (as separation T_1).

In particular, it is also interesting to consider the role of the main Marolf-Ross topology in this setting. So, let M be now a strongly causal spacetime and $\bar{M}$ its Marolf-Ross completion (or any admissible one). For any subset $S \subset \bar{M}$ consider the following subsets of $\bar{M}$:

$$\begin{array}{l} L^+_{IF}(S) = \{(P, F) \in \bar{V} : F \neq \emptyset, F \subset \cup_{(P', F') \in S} F'\}, \\ L^-_{IP}(S) = \{(P, F) \in \bar{V} : P \neq \emptyset, P \subset \cup_{(P', F') \in S} P'\} \end{array}$$

If $L^+_{IF}(S), L^-_{IP}(S)$ are computed in M they yield the closures of $I^{\pm}(S)$. Nevertheless, the restriction $F \neq \emptyset$ in $L^+_{IF}(S)$ (which is clearly necessary, as the empty set is included in the subsequent union of subsets) makes it so that no pair $(P, \emptyset)$ belongs to $L^+_{IF}(S)$. This situation is rectified by introducing two operators Cl_{FB}, Cl_{PB}, the closures in the future and past boundaries:

$$\begin{array}{l} Cl_{FB}(S) = S \cup \{(P, \emptyset) \in \bar{M} : P \in \hat{L}(P_n) \text{ for some sequence } \{(P_n, F_n)\}_n \subset S\}, \\ Cl_{PB}(S) = S \cup \{(\emptyset, F) \in \bar{M} : F \in \check{L}(F_n) \text{ for some sequence } \{(P_n, F_n)\}_n \subset S\}. \end{array}$$

[10]Notice that property (ii) is not fulfilled by Marolf-Ross', (iii) is not fulfilled by Harris', and (iv) is not fulfilled by Harris' and the alternative Marolf-Ross topology.

Then, the Marolf-Ross topology is generated by using as closed subsets:

$$L^+(S) = Cl_{FB}(S \cup L^+_{IF}(S)), \quad L^-(S) = Cl_{PB}(S \cup L^-_{IP}(S)).$$

Even though this choice of topology is natural, the possibility of different alternatives, especially in the definition of the operators Cl_{FB}, Cl_{PB}, was already pointed out by Marolf and Ross. It is also worth pointing out that the definition of these operators in [41] does not use our limit operators $\hat{L}, \check{L}$, even though it becomes equivalent for pairs $(P, \emptyset), (\emptyset, F)$ (see [16]). Very remarkably, the unique differences of limits of sequences between Marolf-Ross and Flores topologies, may occur only when one of the components (P, F) of the candidate to limit is empty; in this case, Marolf-Ross topology may be non-T_1 (a typical example would by Fig. 5); this will be proven in a further study [16]. So, as commented above, Flores' choice fix the best behaved topology, among the admissible ones in the setting of completions.

5.5 Note: timelike boundary and extended causal relation

The extension $\bar{\ll}$ of $\ll$ to the completion $\bar{M}$ is not only natural for the construction of the c-boundary, but also a source of information on the original chronological set. For example, a strongly causal spacetime M, or even just a causal one, is *not* globally hyperbolic if and only if there exists a boundary point $Q = (P, F) \in \partial M$ which satisfies $x \overline{\ll} Q \overline{\ll} y$ for some $x, y \in M$ (see [13, Th. 9.1], and recall [7], [50], [14, Th. 5.11]). This condition is equivalent to $F \neq \emptyset \neq P$, and such pairs constitute the *timelike boundary* $\partial_0 M$. As commented in the Introduction, ∂M splits then in three subsets, $\partial_0 M$, $\partial_+ M$ (composed by pairs $(P, \emptyset)$) and $\partial_- M$ (pairs $(\emptyset, F)$). In the globally hyperbolic case, the spacetime splits smoothly as an orthogonal product $\mathbb{R} \times S$ (see [6]), and the boundary is expected to reflect asymptotic directions, event horizons, and other causal elements. Notice that even in this case ($\partial_0 M = \emptyset$) the parts $\partial_\pm M$ may look like very different; for example, in a standard half-cylinder $(\mathbb{R}^- \times \mathbb{S}^1, -dt^2 + d\theta^2)$, $\partial_- M$ is a point and $\partial_+ M$ a circle $\mathbb{S}^1$.

Up to now, we have considered only the chronological relation $\ll$ and its extension to the boundary $\overline{\ll}$. However, it is also natural to wonder for an extended causal relation and, then for the "lightlike" parts of $\partial_\pm M$. Following [24], there is a way to construct a causal relation $\leq^{\ll}$ from any chronological relation $\ll$, namely:

$$x \leq^{\ll} y \Leftrightarrow I^-(x) \subset I^-(y) \text{ and } I^+(x) \supset I^+(y).$$

For spacetimes, $\leq^{\ll}$ agrees $\leq$ in causally simple spacetimes (but not in $\mathbb{L}^2 \backslash \{0\}$, consider $(-1, -1)$ and $(1, 1)$). So, if we consider the completion $\bar{M}$ of a spacetime M and take the associated causal relation $\leq^{\overline{\ll}}$, this will introduce spurious causal relations in M. Harris [27] also introduced a notion of *chronological set X with only spacelike boundaries*. In the regular case, this comprises two conditions: (a) if $P \in \partial \hat{X}$ then $I^-(P)$ (computed with $\ll^c$ in (3.2)) is not included in $I^-(Q)$ for any $Q \in \hat{X}$, and (b) ∂X is closed (for Harris topology) in $\hat{X}$. But notice that condition (b) is technical, and condition (a) (which surely would mean $P \not\leq^c Q$ for any sensible definition of a extended causal relation $\leq^c$) is very restrictive. So, it does not suggest how to introduce a causal relation in $\bar{M}$. Finally, Marolf and Ross studied some possible alternatives, and the associated problems regarding transitivity and reflexivity [41, Sect. 3.2]; one could add even some more alternatives (recall Remark 5.1). However, as Marolf-Ross pointed out, this problem is not essential at this level. In the case of waves with a 1-dimensional boundary, some properties suggest

that this boundary might be regarded as lightlike. But one can postpone a proper definition until new issues make it necessary.

6 Computation of the boundary of the waves

Next, we will sketch how to compute the c-boundary in the case of wave-type spacetimes. The computation of the boundary for the simplest case, i.e., product spacetimes $\mathbb{L}^1 \times S$ is not trivial [1]. Busemann functions are required [28], and subtleties at the topological level appear [14] (see also [15]). However, the result is very intuitive and, at least at the level of the boundary as a point set, more or less expected. Nevertheless, as Marolf and Ross pointed out, plane waves are a physical example were the necessity of a rigorous definition of the c-boundary is stressed –in particular, the unique guidance to make pairs (P, F) is the abstract Szabados relation. However, in the case of reasonably physical wave-type spacetimes, some of the subtleties of this boundary do not appear. In fact, Marolf-Ross' is also a minimal completion and, thus, the unique admissible one.

6.1 The class of wave-type spacetimes

Next, consider the class of spacetimes, namely, Mp-waves:

$$M = M_0 \times \mathbb{R}^2, \ \ \langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle_0 - F(x, u)\ du^2 - 2\ du\ dv$$

where $(M_0, \langle \cdot, \cdot \rangle_0)$ is a n_0-dimensional Riemannian manifold, (v, u) are the natural coordinates of $\mathbb{R}^2$ and $F : M_0 \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Particular cases are the *pp-waves* (plane fronted waves with parallel rays), where $(M_0, \langle \cdot, \cdot \rangle_0)$ is just $\mathbb{R}^2$ (or, in general, we also consider $\mathbb{R}^{n_0-2}$); the vacuum or *gravitational* pp-waves correspond to $\Delta_x F(x, u) \equiv 0$. Especially, plane waves are characterized as those with $F(x, u)$ a quadratic form in x for each u. For more background on these spacetimes, see [10, 18, 19].

The progress in the computation of the c-boundary is contained in:

- Berenstein and Nastase's seminal work [5] on the conformal boundary for plane waves, under the assumptions: (i) locally symmetric ($F(x, u) \equiv F(x)$) and (ii) conformally flat (the eigenvalues of F are positive and equal).
- Marolf and Ross article [40], under their causal boundary approach (see also [42]). Again, only plane waves are considered, but the assumption on conformal flatness is removed.
- Works by Hubeny, Rangamani and Ross [33, 34, 35]: TIP's and TIF's are computed for some specific pp-wave backgrounds for strings.
- Flores and the author's general framework for the c-boundary in any wave-type spacetime [17]. This includes relevant cases of pp-waves, and will be described below.

6.2 Previous remarks

For the general approach in [17], notice as previous considerations:

(i) The difficult part will be to compute TIPs, TIFs and their common future or pasts, $P, F, \uparrow P, \downarrow F$. In fact, each P will be S-related with at most one F in the interesting cases.

(ii) In principle, one must compute $I^{\pm}(\gamma)$, $\uparrow I^-(\gamma) \downarrow I^+(\gamma)$ for any timelike γ. Nevertheless, it will be enough to consider lightlike curves[11].

(iii) The behaviors of F essential for causality depend on the behavior of $F(x,u)$ for large $|x|$ (where $|x|$ denotes the distance to a fixed point, if M_0 is a Riemannian manifold), [18]. Concretely: (a) F superquadratic in x and $-F$ subquadratic, which implies that M is not distinguishing –the c-boundary is not properly well-defined–, (b) F at most quadratic, i.e. $F(x,u) \leq R_1(u)|x|^2 + R_0(u)$ for continuous functions R_i, which implies that M is strongly causal, and (c) F subquadratic and M_0 complete, which implies that M is globally hyperbolic.

Notice that there exists a critical behavior for the causality of the boundary when F is $x-$quadratic. So, one also defines the behaviors:

- F asymptotically quadratic $R_1^-(u)|x|^2 + R_0^-(u) \leq F(x,u) \leq R_1(u)|x|^2 + R_0(u)$
- In particular, F is λ-asymptotically quadratic ($\lambda > 0$) when :

$$\frac{\lambda^2|x|^2 + R_0^-}{u^2+1} \leq F(x,u) \leq R_1(u)|x|^2 + R_0(u) \qquad \forall (x,u) \in M \times \mathbb{R}$$

(For the properties of the c-boundary, this behavior will be enough in some M_0-direction.)

6.3 Ingredients for the general computation

The geometric and analytic tools necessary for the computation are essentially three:

1.- Functional approach associated to an arrival time. For any $z_0 = (x_0, u_0, v_0) \in M$ parameterize with u each lightlike curve (which is not the integral curve of ∂_v) starting at z_0: $\gamma(u) = (x(u), u, v(u)), u \in I \subset \mathbb{R}$. As $\langle \gamma', \gamma' \rangle \equiv 0$ one has:

$$v(u) = v_0 + \frac{1}{2}\int_{u_0}^{u} (|\dot{x}(\sigma)|^2 - F(x(\sigma), \sigma))d\sigma, \quad \forall u \in I. \tag{6.1}$$

Consider the lower extreme of $I^+(z_0) \cap L_{(x_0,u_0)}$, where $L_{(x_0,u_0)} = \{(x_0, u_0, v) : v \in \mathbb{R}\}$. This extreme (minus v_0) is the "arrival time" from z_0 to the lightlike line $L_{(x_0,u_0)}$. It is equal to the infimum of all the possible $v(u)$ which can be obtained from (6.1).

Now, consider the set $\mathcal{C}(\equiv \mathcal{C}(x_0, x_1; |\Delta u|))$ of curves $x : [0, |\Delta u|] \to M_0$ joining x_0 with some $x_1 \in M_0$. Then, $I^+(z_0)$ is controlled by the infimum of the "arrival time functional":

$$\mathcal{J}_{u_0}^{\Delta u} : \mathcal{C} \to \mathbb{R}, \quad \mathcal{J}_{u_0}^{\Delta u}(y) = \frac{1}{2}\int_0^{|\Delta u|} (|\dot{y}(s)|^2 - F(y(s), u_\nu(s)))ds.$$

where $u_\nu(s) = u_0 + \nu(\Delta u)s, \nu = +1$ (and analogously $I^-(z_0)$).

Notice that $\mathcal{J}_{u_0}^{\Delta u}$ is, in fact, a Lagrangian. As we are really interested in, say, $P = I^-(\gamma)$ and $\uparrow P = \uparrow I^-(\gamma)$, one has to study limits such as:

$$\mathrm{Inf}(\mathcal{J}_{u_0}^{\Delta u}) \;\text{ on } \mathcal{C}(x_0, x_\Delta; |\Delta u|) \quad \text{with } x_\Delta = x(u_\Delta), \text{ and } u_\Delta = u_0 + \Delta u \nearrow u_\infty$$

[11] However, a subtlety appears: in general, the sets type $I^{\pm}(\gamma)$ are equal for timelike and lightlike curves, but those type $\uparrow I^-(\gamma)$, $\downarrow I^+(\gamma)$ are not.

Abstract technical conditions on $\mathcal{J}_{u_0}^{\Delta u}$ (implied by the suitable asymptotic behaviors of F above) control when the TIPs P "collapse" to a 1-dimensional boundary –as in the examples provided by Berenstein and Nastase, and subsequent studies.

2.- Busemann type functions. In order to check when pairs P, F are S-related, TIP's and TIF's as well as their common futures/pasts must be computed explicitly.

In standard static spacetimes (conformal to products $\mathbb{L}^1 \times S$), certain Busemann-type functions, constructed from curves "in the spatial part", were useful to compute TIP's [28]. Here, from curves in M_0 one constructs: (a) an adapted version b^- of such Busemann functions in order to compute TIP's and TIF's, and (b) a more refined version b^+ in order to compute common future/pasts.

3.- Sturm-Liouville theory. In order to study accurately the limit cases of quadratic Mp-waves (some of them of special physical interest), the Euler-Lagrange equation for the functional $\mathcal{J}_{u_0}^{\Delta u}$ must be analyzed. In fact, not only this equation must be studied but also the variation of the solutions with $u_0 + \Delta u \nearrow u_\infty$. This yields an associate Sturm-Liouville problem, especially interesting for Mp-waves with critical behavior.

6.4 Table of results

With all the previous elements, the results can be summarized as follows. We refer to [17] for detailed proofs and explanations.

Qualitative F	Causality	Boundary ∂M	Some examples
F superquad. $-F$ at most quad.	No distin- guishing	No boundary	pp-waves yielding Sine-Gordon string
At most quad. F (resp. $\lvert F\rvert$)	Strongly causal	In principle, computable	all below
λ-asymp. quad. $\lambda > 1/2$	Strongly causal	1-dimension, [lightlike]	plane waves with some eigenv. $\mu_1 \geq \lambda^2/(1+u^2)$ for $\lvert u\rvert$ large
λ-asymp. quad. $\lambda \leq 1/2$	Strongly causal	Critical	pp-wave with $F(x,u) = \lambda^2 x^2/(1+u)^2$ (for $u > 0$)
Subquadratic	Globally hyperbolic	No identif. in $\hat{\partial}M, \check{\partial}M$ Expected higher dim.	(1) $\mathbb{L}^n$ and static type waves (2) plane waves with $-F$ quadratic

7 Conclusion

Finally, let us summarize both, our proposal for the c-boundary, and how it must be computed ideally –this purges all the subtleties and approaches reviewed.

The c-boundary makes sense at least for any strongly causal spacetime M –where all the desired properties for the topology will hold surely. Now:

1. In order to obtain ∂M as a point set, compute all the TIP's and TIF's, i.e., $\hat{M}, \check{M}$. Then, consider all the pairs of type: (a) (P, F), such that $P \sim_S F$ according to S-relation (3.1), and (b) $(P, \emptyset), (\emptyset, F)$ when P or F are not S-related with anything (for $P \in \hat{M}, F \in \check{M}$ always).

   Typically, each P or F will be S-related to at most one element, which will be also rather intuitive (for example, in standard static spacetimes [1], or when a suitable conformal boundary exists [16]). Nevertheless, wave-type spacetimes provide examples where the pairing (P, F) is not by any means evident. And one can construct relatively simple examples (open subsets of $\mathbb{L}^n$, or standard stationary spacetimes [15]), where a TIP or TIF is S-related to more than one element.

2. Extend the chronological relation to $\bar{M}$ by using (5.2) –of course, regarding any point in M as a pair $(I^-(p), I^+(p))$.

   The timelike part $\partial_0 M$ of the boundary ∂M is composed of the pairs (P, F) with $P \neq \emptyset \neq F$. It is empty if and only if M is globally hyperbolic. The points with $F = \emptyset$ (resp. $P = \emptyset$) define the future infinite $\partial_+ M$ (resp. the past infinity $\partial_- M$) of M.

3. The topology in $\bar{M}$ is defined according to Definition 5.8 (see also (5.4), (5.3), (3.4)). For a sequence $\{Q_n = (P_n, F_n)\}_n$ in $\bar{M}$ with $LI\{P_n\} = LS\{P_n\} =: P_\infty, LI\{F_n\} = LS\{F_n\} =: F_\infty$, this will mean the following. A point $Q = (P, F) \in \bar{M}$ lies in the closure of $\{Q_n\}_n$ iff either $P = P_\infty$ and $F = F_\infty$ or, at least, P, F are maximal among all the (past or future) indecomposable sets included in P_∞, F_∞, respectively –that is: $P \subset P_\infty, F \subset F_\infty$ and no $P' \in \hat{M}, F' \in \check{M}$ satisfy either $P \subsetneq P' \subset P_\infty$ or $F \subsetneq F' \subset F_\infty$.

   Then, all the reasonable properties of a boundary hold for ∂M (it is closed in M, points of the boundary are T_2-separated of the points of the spacetime M, etc.). In particular, all the points in ∂M are T_1 separated, and the cases where they are not T_2 separated are intuitively clear and acceptable.

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